

The continuum limit of a 4-dimensional causal set scalar d'Alembertian

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The continuum limit of a 4-dimensional, discrete d'Alembertian operator for scalar fields on causal sets is studied. The continuum limit of the mean of this operator in the Poisson point process in 4-dimensional Minkowski spacetime is shown to be the usual continuum scalar d'Alembertian \square . It is shown that the mean is close to the limit when there exists a frame in which the scalar field is slowly varying on the scale set by the density of the Poisson process. The continuum limit of the mean of the causal set d'Alembertian in 4-dimensional curved spacetime is shown to equal $\square - \frac{1}{2}R$, where R is the Ricci scalar, under certain conditions on the spacetime and the scalar field.

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I. INTRODUCTION

Causal sets are discrete spacetimes proposed as the histories in a sum-over-histories approach to quantum gravity [1]. They can also play a *phenomenological* role as models of spacetimes with no structure on scales smaller than the Planck scale and in which there is no preferred frame. The existence of a physical Planck scale cutoff is indicated from many different directions in physics, most notably and convincingly by the value of the black hole entropy [2]. The feature that distinguishes causal sets from other models with Planck scale uncertainty in spacetime structure is Lorentz invariance. Thus, causal sets embody the breakdown of continuum spacetime at the Planck scale that is widely expected in quantum gravity whilst adhering to a physical symmetry that has been the basis for great progress in fundamental physics.

That discreteness can be compatible with Lorentz invariance is welcome for workers guided by the unity of physics. However, there is a price: the discreteness and Lorentz invariance of causal sets together result in a radical nonlocality [1, 3, 4]. This nonlocality, were it incorrigible, could prevent causal sets from being useful phenomenologically and threaten to derail the causal set programme for quantum gravity. Thus, evidence that the nonlocality of Lorentz invariant discrete structure can be tamed [5, 6] has been important to the causal set programme. More recently further evidence was provided by the discovery of a quasi-local, discrete scalar d'Alembertian operator, $B^{(2)}$ for fields on causal sets well-approximated by 2 dimensional Minkowski spacetime [7, 8]. This operator tends to the exact continuum 2 dimensional flat scalar d'Alembertian in the continuum limit [9]. In [10] this work was extended to 4 dimensions with the introduction of an analogous operator $B^{(4)}$. There it was claimed that for both $d = 2$ and $d = 4$, when $B^{(d)}$ is applied to scalar fields on causal sets which are approximated by d -dimensional Lorentzian spacetimes, its mean tends in the continuum limit to the curved space operator, $\square - \frac{1}{2}R$, where \square is the curved spacetime scalar d'Alembertian and R is the Ricci scalar curvature. In this paper we will prove this result in four dimensions under certain conditions.

Although these operators (and their generalisations to any dimension [11–13]) do indeed tame the radical nonlocality referred to above, they do not eliminate it altogether, in the sense that for a finite discreteness scale, l , the nonlocality survives on scales of order l . This remnant of nonlocality is manifest in the dynamics of (scalar) fields on spacetime, which becomes nonlocal. Nonlocal dynamics of exactly this form was recently used in the construction of scalar nonlocal quantum field theories in [14] and [15], potentially leading to novel and interesting phenomenology.

Recall that a causal set (or causet) is a locally finite partial order, (\mathcal{C}, \preceq) . Local finiteness is the condition that the cardinality of any *order interval* is finite, where the (inclusive) order interval between a pair of elements $y \preceq x$ is defined to be $I(x, y) := \{z \in \mathcal{C} \mid y \preceq z \preceq x\}$. We write $x \prec y$ when $x \preceq y$ and $x \neq y$. We call a relation $x \prec y$ a *link* if the order interval $I(x, y)$ contains only x and y . We denote by $|\cdot|$ the cardinality of a set and $n(x, y) := |I(x, y)| - 2$.

Given a point $x \in \mathcal{C}$ we define the set of all its past nearest neighbours to be

$$L_1(x) := \{y \in \mathcal{C} \mid y \prec x, n(x, y) = 0\}. \quad (1)$$

We refer to this set of elements as the first *past layer*. We can generalise this by defining the

sets of next nearest neighbours, L_2 , next next nearest neighbours, L_3 , and so on. In general the i -th past layer is defined as

$$L_i(x) := \{y \in \mathcal{C} \mid y \prec x \text{ and } n(x, y) = i - 1\}. \quad (2)$$

Consider the discrete retarded operator B , on a causet \mathcal{C} , defined as follows [10]. If $\phi : \mathcal{C} \rightarrow \mathbb{R}$ is a scalar field, then

$$B\phi(x) := \frac{4}{\sqrt{6}l^2} \left[-\phi(x) + \left(\sum_{y \in L_1(x)} -9 \sum_{y \in L_2(x)} + 16 \sum_{y \in L_3(x)} - 8 \sum_{y \in L_4(x)} \right) \phi(y) \right], \quad (3)$$

where l is a length (the analogue of the lattice spacing).

B is defined on scalar fields on any causal set but is particularly relevant for causal sets that are well-approximated by a four dimensional Lorentzian manifold, (\mathcal{M}, g) . A causet, (\mathcal{C}, \preceq) is well approximated by (\mathcal{M}, g) if there exists a *faithful embedding* of \mathcal{C} into \mathcal{M} in which the causal order of the embedded elements respects the order of \mathcal{C} and in which the number of causet elements embedded in any sufficiently nice, large region of \mathcal{M} approximates the spacetime volume of that region in fundamental units. These manifold-like, faithfully embeddable causets are ones that could have arisen with relatively high probability in the random process of *sprinkling* into (\mathcal{M}, g) . Sprinkling is a Poisson process of selecting points in \mathcal{M} with density ρ so that the expected number of points sprinkled in a region of spacetime volume V is ρV . To do justice to our expectations for quantum gravity, the density $\rho = l^{-4}$, where l is the fundamental length scale of the order of the Planck length. The probability for sprinkling m elements into a region of volume V is

$$P(m) = \frac{(\rho V)^m e^{-\rho V}}{m!}, \quad (4)$$

This process generates a causet, \mathcal{C} whose elements are the sprinkled points and whose order, \preceq is that induced by the manifold's causal order restricted to the sprinkled points.

Let ϕ be a real test field of compact support on \mathcal{M} and $x \in \mathcal{M}$. If we sprinkle \mathcal{M} at density ρ , include x in the resulting causet, \mathcal{C} , then $L_1(x) \subset \mathcal{C}$ will be a set whose elements lie in the causal past of x , $J^-(x)$, and hug the boundary of $J^-(x)$. Their locus is roughly the hyperboloid which lies one Planck unit of proper time to the past of x . The elements of $L_2(x)$ will also be distributed down the inside of the boundary of $J^-(x)$, just inside layer 1, and so on. The operator B can be applied, at point x , to the field ϕ restricted to the

sprinkled causet: $B\phi(x)$ looks highly nonlocal, involving the value of ϕ at enormous numbers of points outside any fixed neighbourhood of x .

The sprinkling process at density ρ produces, for each point x of \mathcal{M} , a random variable whose value is $B\phi(x)$ on the realisation \mathcal{C} of the process. The expectation value of this random variable is given by the spacetime integral

$$\bar{B}\phi(x) := \mathbb{E}(B\phi(x)) = \frac{4\sqrt{\rho}}{\sqrt{6}} \left[-\phi(x) + \rho \int_{y \in J^-(x)} d^4y \sqrt{-g} \phi(y) e^{-\xi} (1 - 9\xi + 8\xi^2 - \frac{4}{3}\xi^3) \right], \quad (5)$$

where $\xi := \rho V(y)$ and $V(y)$ is the volume of the causal interval between x and y .

We can see that the integrand is suppressed wherever ξ is large, *i.e.* wherever the spacetime volume of the causal interval between x and y is larger than a few Planck volumes. However, ξ is small in the part of the region of integration close to the boundary of $J^-(x)$ and, by itself, the exponential factor in the integrand cannot provide the suppression required to give a value that is approximately a local quantity at x . In the following we will show that, for large enough ρ , $\bar{B}\phi(x)$ is effectively local and is dominated by contributions from a neighbourhood of x : the contributions from far down the boundary of $J^-(x)$ cancel out. Indeed, we will show

$$\lim_{\rho \rightarrow \infty} \bar{B}\phi(x) = \square\phi(x) - \frac{1}{2}R(x)\phi(x) \quad (6)$$

under certain assumptions about the support of ϕ in $(\mathcal{M}^{(4)}, g)$. We use the conventions of Hawking and Ellis [16].

II. MINKOWSKI SPACETIME

We consider the simpler case of a sprinkling in Minkowski spacetime for which we will show that $\lim_{\rho \rightarrow \infty} \bar{B}\phi(x) = \square\phi(x)$. Although this is a special case of the curved space result, it is useful to see this simpler proof first as it will provide the basic framework on which the curved spacetime calculation is built. We will also be able to estimate the corrections to the limiting value, something that will turn out to be much harder to do in the curved case.

Choose x as the origin of cartesian coordinates $\{y^\mu\}$ and in that frame define the usual spatial polar coordinates: $r = \sqrt{\sum_{i=1}^3 (y^i)^2}$, θ and φ . Null radial coordinates (past pointing) are defined by $u = \frac{1}{\sqrt{2}}(-t - r)$ and $v = \frac{1}{\sqrt{2}}(-t + r)$ where $t = y^0$. The volume, $V(y)$, of the causal interval between point y with cartesian coordinates $\{y^\mu\}$ and the origin is $V(y) = \frac{\pi}{6}u^2v^2$.

Let us take the region of integration \mathcal{W} to be the portion of the causal past of the origin for which $v \leq L$, where L is large enough that the support of ϕ is contained in \mathcal{W} . \mathcal{W} can be split into 3 parts:

$$W_1 := \{y \in \mathcal{W} \mid 0 \leq u \leq v \leq a\} \quad (7)$$

$$W_2 := \{y \in \mathcal{W} \mid a \leq v \leq L, 0 \leq u \leq \frac{a^2}{v}\} \quad (8)$$

$$W_3 := \mathcal{W} \setminus (W_1 \cup W_2), \quad (9)$$

where $a > 0$ is chosen small enough that the expansions of ϕ used in the following calculation are valid. W_1 is a neighbourhood of the origin, W_2 is a neighbourhood of the past lightcone and bounded away from the origin and W_3 is a subset of the interior of the causal past that is bounded away from the lightcone, see Figure 1.

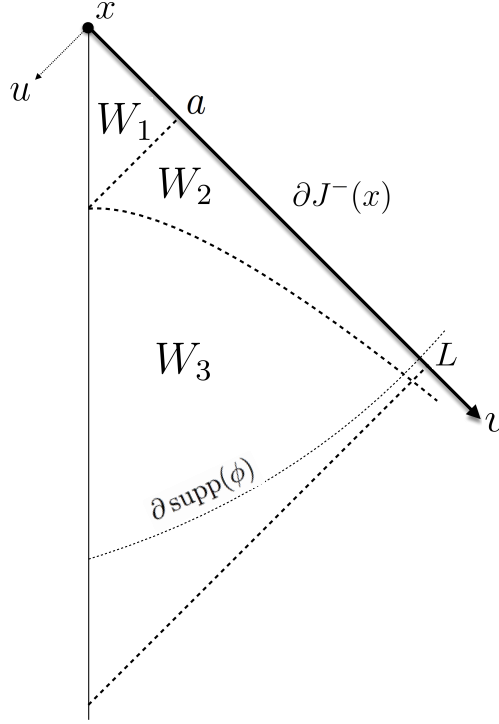


FIG. 1: Partition of \mathcal{W} into regions W_1 , W_2 and W_3 in $(t-r)$ -plane

Let

$$I_i = \int_{W_i} d^4y \, \phi(y) e^{-\rho V(y)} (1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3), \quad (10)$$

for $i = 1, 2, 3$ so that

$$\bar{B}\phi(x) := \frac{4\sqrt{\rho}}{\sqrt{6}} \left[-\phi(x) + \rho(I_1 + I_2 + I_3) \right]. \quad (11)$$

A. The “deep chronological past,” W_3

We first consider first I_3 . $V(y)$ is bounded away from zero in W_3 , indeed $V(y) \geq V_{\min} = \frac{\pi}{6}a^4$ so

$$|I_3| \leq e^{-\rho V_{\min}} \int_{W_3} d^4y \left| \phi(y)(1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3) \right| \quad (12)$$

which tends to zero faster than any power of ρ^{-1} as $\rho \rightarrow \infty$. In what follows, we will often write, “up to exponentially small terms” which means we are neglecting terms like I_3 .

B. Down the light cone, W_2

Consider now I_2 . Note first that

$$e^{-\xi}(1 - 9\xi + 8\xi^2 - \frac{4}{3}\xi^3) = \hat{\mathcal{O}}e^{-\xi} \quad (13)$$

where

$$\hat{\mathcal{O}} := \frac{4}{3}(H + \frac{1}{2})(H + 1)(H + \frac{3}{2}) \quad (14)$$

$$= 1 + 9H_1 + 8H_2 + \frac{4}{3}H_3 \quad (15)$$

and

$$H_n := \rho^n \frac{\partial^n}{\partial \rho^n} \quad \text{and} \quad H := H_1. \quad (16)$$

The integral we are evaluating can be rewritten as

$$I_2 = \int d\Omega_2 \hat{\mathcal{O}} \int_a^L dv \int_0^{\frac{a^2}{v}} du \frac{1}{2}(v - u)^2 \phi(y) e^{-\sigma u^2 v^2}, \quad (17)$$

where $\int d\Omega_2 = \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\varphi$. We have absorbed a factor of $\pi/6$ into $\sigma = \pi\rho/6$, for convenience. Note that $\hat{\mathcal{O}}$ is unchanged by that manoeuvre:

$$H = \rho \frac{\partial}{\partial \rho} = \sigma \frac{\partial}{\partial \sigma}. \quad (18)$$

We will see that $I_2 = O(\rho^{-2})$ (equivalently $O(\sigma^{-2})$) and so, when multiplied by $\rho^{3/2}$, makes no contribution to the limit of $\bar{B}\phi(x)$. This can be understood by noticing that $\hat{\mathcal{O}}$ annihilates $\rho^{-1/2}$, ρ^{-1} and $\rho^{-3/2}$. If the function of ρ on which $\hat{\mathcal{O}}$ acts is well-behaved enough as $\rho \rightarrow \infty$ to be equal to a power series expansion in $\rho^{-1/2}$, then application of $\hat{\mathcal{O}}$ will eliminate all the terms that would not – after multiplication by $\rho^{3/2}$ – tend to zero. Another way to understand the result is to notice that the integral over W_2 involves an integral over the null coordinate u , transverse to the light cone. If the range of the u integration is small enough – if W_2 is close enough to the light cone – then ϕ will be approximately constant in u at fixed values of the other coordinates. The integration over u for *constant* ϕ is

$$\int_0^{\frac{a^2}{v}} du \left(1 - 9\sigma v^2 u^2 + 8(\sigma v^2 u^2)^2 - \frac{4}{3}(\sigma v^2 u^2)^3 \right) (v - u)^2 e^{-\sigma v^2 u^2}. \quad (19)$$

The oscillating integrand gives exactly the right amount of cancellation so that the value of the integral is suppressed by $\exp(-\sigma a^4)$. This suggests that the leading, finite ρ corrections to the limit is set by the u derivatives of ϕ at $u = 0$, and this turns out to be the case.

We assume that a is chosen small enough and that ϕ is differentiable enough that throughout W_2 , $\phi(y)$ can be expanded in the transverse coordinate u :

$$\phi(y) = \phi|_{u=0} + u\phi_{,u}|_{u=0} + \frac{1}{2!}u^2\phi_{,uu}|_{u=0} + \frac{1}{3!}u^3\Phi(y), \quad (20)$$

where $\Phi(y)$ is a continuous function. Let $I_2 = I_{2,0} + I_{2,1} + I_{2,2} + I_{2,3}$ where

$$I_{2,i} = \int d\Omega_2 \int_a^L dv F_i \int_0^{\frac{a^2}{v}} du \frac{(v-u)^2}{2} u^i (1 - 9\sigma u^2 v^2 + 8\sigma^2 u^4 v^4 - \frac{4}{3}\sigma^3 u^6 v^6) e^{-\sigma u^2 v^2}, \quad (21)$$

and $F_i = \phi|_{u=0}$, $\phi_{,u}|_{u=0}$ and $\frac{1}{2}\phi_{,uu}|_{u=0}$ for $i = 0, 1$ and 2 respectively, and

$$I_{2,3} = \int d\Omega_2 \int_a^L dv \int_0^{\frac{a^2}{v}} du \frac{\Phi(y)}{3!} \frac{(v-u)^2}{2} u^3 (1 - 9\sigma u^2 v^2 + 8\sigma^2 u^4 v^4 - \frac{4}{3}\sigma^3 u^6 v^6) e^{-\sigma u^2 v^2}. \quad (22)$$

For $I_{2,i}$, $i = 0, 1, 2$, the u and v integrations can be done explicitly and the remaining integral over the angular coordinates can be bounded by bounding F_i by its uniform norm over the light cone $u = 0$. We find that, up to exponentially small contributions, $I_{2,0}$ vanishes and

$$|I_{2,1}| \leq \frac{\pi \|\phi_{,u}\|_{LC}}{3\sigma^2} \left(\frac{1}{a^3} - \frac{1}{L^3} \right) \quad (23)$$

$$|I_{2,2}| \leq \frac{\pi \|\phi_{,uu}\|_{LC}}{\sigma^2} \left(\frac{1}{a^2} - \frac{1}{L^2} \right) + O\left(\frac{1}{\sigma^{5/2}}\right), \quad (24)$$

where $\|\cdot\|_{LC}$ denotes the uniform norm over the light cone $u = 0$ in W_2 .

Finally we must bound $I_{2,3}$. Since Φ is a function of u , each of the integrals corresponding to the terms in the polynomial in the integrand (22) must be individually bounded: the cancellation between the terms that happens for $I_{2,i}$, $i = 0, 1, 2$ cannot help here. The value of Φ is bounded by its uniform norm in W_2 and, by Taylor's theorem, this is less than or equal to the uniform norm of the third u -derivative of ϕ in W_2 . We find

$$|I_{2,3}| \leq \frac{\pi \|\phi_{,uuu}\|_2}{6\sigma^2} \left(\frac{1}{a} - \frac{1}{L} \right) + O\left(\frac{1}{\sigma^{5/2}} \right), \quad (25)$$

where $\|\cdot\|_2$ is the uniform norm in W_2 . The key here is that ϕ has been expanded in u far enough that the power of u in the factor u^3 in (22) is high enough for the u integration to bring down enough powers of σ^{-1} . We will see the same thing happening in the integral over region W_1 and again in the curved space case.

Multiplying I_2 by $\rho^{3/2}$, we see that the contribution to $\bar{B}\phi(x)$ from the region W_2 tends to zero in the limit and the leading corrections go like $\rho^{-1/2}$ and are proportional to the u -derivatives of ϕ on and close to the light cone.

C. The near region, W_1

Now consider

$$I_1 = \hat{\mathcal{O}} \int_{W_1} d^4y \phi(y) e^{-\rho V(y)} \quad (26)$$

$$= \hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{1}{2} (v-u)^2 \phi(y) e^{-\rho V(y)}. \quad (27)$$

We assume we can expand the field in W_1 ,

$$\phi(y) = \phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2} y^\mu y^\nu \phi_{,\mu\nu}(0) + y^\mu y^\nu y^\alpha \psi_{\mu\nu\alpha}(y), \quad (28)$$

where $\psi_{\mu\nu\alpha}(y)$ are C^3 -functions of the y^μ (they are not components of a tensor, the indices just label the functions). The first two terms of the above expansion of ϕ contribute to I_1

$$\hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} \phi(0) e^{-\rho \frac{\pi}{6} u^2 v^2} \quad (29)$$

$$= \frac{1}{\rho} (1 - e^{-\frac{\pi}{6} \rho a^4}) \phi(0), \quad (30)$$

and

$$\hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} y^\mu \phi_{,\mu}(0) e^{-\rho \frac{\pi}{6} u^2 v^2} \quad (31)$$

$$= \left(-\frac{6\sqrt{2}}{a^3 \pi \rho^2} + \left(\frac{\sqrt{2}a}{\rho} + \frac{6\sqrt{2}}{a^3 \pi \rho^2} \right) e^{-\frac{\pi}{6} \rho a^4} \right) \phi_{,t}(0). \quad (32)$$

The first term (30) cancels with the term $\phi(x)$ in the expression for $\bar{B}(x)$ (5), while the second contributes nothing in the limit $\rho \rightarrow \infty$. The leading correction at finite ρ behaves as

$$\frac{l^2}{a^3} \phi_{,t}(0). \quad (33)$$

The third term of the expansion of ϕ (28) is of most interest to us: it contributes to $\bar{B}(x)$

$$\frac{4}{\sqrt{6}} \rho^{\frac{3}{2}} \hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{1}{2} (v-u)^2 y^\mu y^\nu \phi_{,\mu\nu}(0) e^{-\rho \frac{\pi}{6} u^2 v^2} \quad (34)$$

$$= \square \phi(0) - \frac{4\sqrt{6}}{a^2 \pi \sqrt{\rho}} \phi_{,ii}(0) + \frac{9}{a^4 \pi \rho} (\phi_{,ii}(0) + 3\phi_{,tt}(0)) \quad (35)$$

up to exponentially small terms (there is a sum on i implied in $\phi_{,ii}$). The leading correction at finite ρ is, up to a factor of order one,

$$\frac{l^2}{a^2} \phi_{,ii}(0). \quad (36)$$

Finally we need to show that the integral

$$\hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} y^\mu y^\nu y^\alpha \psi_{\mu\nu\alpha}(y) e^{-\rho \frac{\pi}{6} u^2 v^2} \quad (37)$$

does not contribute in the limit. Each y^μ is equal to t , $r \cos \theta$, $r \sin \theta \cos \varphi$ or $r \sin \theta \sin \varphi$. At the end of the calculation, the uniform norm of the integrand over the region of integration will be used to bound the integral, and the factors of $\cos \theta$ *etc.* will make no difference to the result and we can drop them now, for convenience. We therefore need to show that each integral of the form

$$\int d\Omega_2 \hat{\mathcal{O}} \int_0^a dv \int_0^v du (v-u)^2 u^m v^n \psi e^{-\sigma u^2 v^2}, \quad m+n=3, \quad (38)$$

tends to zero faster than $\rho^{-3/2}$, where ψ stands for one of the $\psi_{\mu\nu\alpha}(y)$ and is a function of u, v, θ and ϕ , and, again, for convenience we have defined $\sigma = \pi \rho / 6$.

Leaving the integration over the angles for later, consider

$$K_{m,n} := \hat{\mathcal{O}} \int_0^a dv \int_0^v du (v-u)^2 u^m v^n \psi e^{-\sigma u^2 v^2}, \quad m+n=3. \quad (39)$$

Note first that

$$e^{-\sigma u^2 v^2} = \frac{\sqrt{\pi}}{2v} \frac{\partial}{\partial u} \frac{\operatorname{erf}(\sqrt{\sigma} uv)}{\sqrt{\sigma}}. \quad (40)$$

Using this identity and integrating (39) by parts in u we find

$$K_{m,n} = -\hat{\mathcal{O}} \int_0^a dv \int_0^v du \frac{\partial}{\partial u} ((v-u)^2 u^m v^n \psi) \frac{\sqrt{\pi}}{2v} \frac{\operatorname{erf}(\sqrt{\sigma} uv)}{\sqrt{\sigma}}, \quad (41)$$

since the boundary terms vanish. The following identity

$$\hat{\mathcal{O}} \left(\frac{\operatorname{erf}(\sqrt{\sigma} uv)}{\sqrt{\sigma}} \right) = \frac{2}{\sqrt{\pi}} uv \hat{\mathcal{P}} e^{-\sigma u^2 v^2} \quad (42)$$

where $\hat{\mathcal{P}} = \frac{2}{3}(H+1)(H+\frac{3}{2})$ allows one to rewrite the integral as

$$K_{m,n} = -\hat{\mathcal{P}} \int_0^a dv \int_0^v du \frac{\partial}{\partial u} ((v-u)^2 u^m v^n \psi) u e^{-\sigma u^2 v^2}. \quad (43)$$

Integrating by parts again in u we find

$$-\hat{\mathcal{P}} \int_0^a dv \int_0^v du \frac{\partial^2}{\partial u^2} ((v-u)^2 u^m v^n \psi) \frac{e^{-\sigma u^2 v^2}}{2\sigma v^2}. \quad (44)$$

Using (40) and integrating by parts in u again we find

$$K_{m,n} = \hat{\mathcal{P}} \int_0^a dv \left\{ -\frac{\sqrt{\pi}}{2} \frac{\operatorname{erf}(\sqrt{\sigma} v^2)}{\sigma^{3/2}} v^{m+n} \tilde{\psi} + \int_0^v du \frac{\sqrt{\pi}}{4v^3} \frac{\partial^3}{\partial u^3} ((v-u)^2 u^m v^n \psi) \frac{\operatorname{erf}(\sqrt{\sigma} uv)}{\sigma^{3/2}} \right\}. \quad (45)$$

where $\tilde{\psi} := \tilde{\psi}(v, \theta, \phi) = \psi|_{u=v}$. Using the following identity

$$\hat{\mathcal{P}} \left(\frac{\operatorname{erf}(\sqrt{\sigma} z)}{\sigma^{3/2}} \right) = -\frac{2}{3\sqrt{\pi}} z^3 e^{-\sigma z^2}, \quad (46)$$

gives

$$K_{m,n} = -\frac{1}{6} \int_0^a dv \int_0^v du u^3 \frac{\partial^3}{\partial u^3} ((v-u)^2 u^m v^n \psi) e^{-\sigma u^2 v^2} + \frac{1}{3} \int_0^a dv \tilde{\psi} v^{6+m+n} e^{-\sigma v^4}. \quad (47)$$

Including now the integration over angles, the contribution of the second term of (47) to I_1 , is bounded by

$$\int d\Omega_2 \left| \frac{1}{3} \int_0^a dv \tilde{\psi} v^{6+m+n} e^{-\sigma v^4} \right| \leq 4\pi \frac{\|\tilde{\psi}\|_1}{\sigma^{5/2}} \frac{\Gamma(5/2)}{12} \quad (48)$$

up to exponentially small terms, where $\|\cdot\|_1$ is the uniform norm over region W_1 , we have used that $m + n = 3$, and we recall that $\sigma = \pi\rho/6$. When multiplied by $\rho^{3/2}$ this term therefore gives a leading correction of $O(\rho^{-1})$.

Consider now the bulk term in (47), and denote by $u^i v^j$, $i + j = m + n + 2$, a generic term in $(v - u)^2 u^m v^n$. Then

$$u^3 \frac{\partial^3}{\partial u^3} (u^i v^j \psi) = i(i-1)(i-2)u^i v^j \psi + 3i(i-1)u^{i+1} v^j \psi' + 3iu^{i+2} v^j \psi'' + u^{i+3} v^j \psi''', \quad (49)$$

where $'$ denotes differentiation with respect to u . We can write any such term as $u^{i+k} v^j \psi^{(k)}$ where $\psi^{(n)} = \frac{\partial^n}{\partial u^n} \psi$, $i \geq 3 - k$ and $k = 0, 1, 2, 3$. Then, the contribution of each of these terms to I_1 is bounded – up to exponentially small terms – by

$$\frac{\|\psi^{(k)}\|_1}{2(i+k-j)} \left(\frac{1}{\sigma^{\frac{i+j+k+2}{4}}} \Gamma\left(\frac{i+j+k+2}{4}\right) - \frac{a^{j-i-k}}{\sigma^{\frac{i+k+1}{2}}} \Gamma\left(\frac{i+k+1}{2}\right) \right) \quad (50)$$

for $i + k \neq j$, and

$$\frac{\|\psi^{(k)}\|_1}{8\sigma^{\frac{i+k+1}{2}}} \Gamma\left(\frac{i+k+1}{2}\right) \left(\log(\sigma a^4) - \frac{\Gamma'(\frac{i+k+1}{2})}{\Gamma(\frac{i+k+1}{2})} \right) \quad (51)$$

for $i + k = j$. The leading order, finite ρ correction occurs when $i + j = 5$, $k = 0$ and (after being multiplied by $\rho^{3/2}$) is $O(\rho^{-1/4})$. The next correction, which we'll be interested in in the next section, occurs when $i + k = 3$. The angular integration $\int d\Omega$ just contributes an overall factor of 4π .

All contributions to $\bar{B}\phi(x)$ have now been accounted for and we see that its limit is $\square\phi(x)$ as $\rho \rightarrow \infty$.

D. Finite ρ corrections

The correct continuum value for the limit of the mean is a good sign that the non locality of causal sets can be tamed. However, for causal sets the important question is how $\bar{B}\phi(x)$ behaves when the discreteness length l is of order the Planck length so that $\rho = l^{-4}$ is large but finite.

The above calculations show that $\bar{B}\phi$ is a good approximation to $\square\phi$ at finite ρ whenever there exists a coordinate frame and a length scale a such that $\rho a^4 \gg 1$, *i.e.* $l \ll a$ — so that the “exponentially small terms” referred to in the calculations above are indeed small

— and such that the following conditions on the derivatives of ϕ in that frame hold:

$$\frac{l^2}{a^3} \|\phi_{,u}\|_{LC}, \frac{l^2}{a^2} \|\phi_{,uu}\|_{LC}, \frac{l^2}{a} \|\phi_{,uuu}\|_{LC} \ll \square\phi(0), \quad (52)$$

from down the light cone and

$$\frac{l^2}{a^3} \phi_{,t}(0), \frac{l^2}{a^2} \phi_{,ii}(0), l \|\psi\|_1, l^2 a^{n-3} \|\psi^{(n-2)}\|_1, l^2 \ln(a/l) \|\psi^{(1)}\|_1 \ll \square\phi(0) \quad (53)$$

$n = 2, 3, 4, 5$, where $\psi^{(n)}$ denotes the n -th u -derivative of ψ , from the near region.

We see that if a frame exists in which ϕ is slowly varying on the discreteness scale close to x and close to the light cone then $\bar{B}\phi$ is a good approximation to $\square\phi$. This is quite a strong condition however and it is possible that it can be weakened. For example, suppose there is no global frame in which ϕ is slowly varying on the discreteness scale, but the causal past of x close to the lightcone can be covered by patches in each of which there are coordinates in which the condition holds. It might be possible to show that the contribution from each patch vanishes in the limit and thus prove a more powerful result. We will discuss this possibility further in the final section.

III. CURVED SPACETIME

We assume again that the field ϕ is of compact support and we will again split the region of integration $J^-(x) \cap \text{Support}(\phi)$ into three parts: the deep chronological past, W_3 , bounded away from $\partial J^-(x)$; W_2 , a neighbourhood of $\partial J^-(x)$ bounded away from x ; and the near region, W_1 , a neighbourhood of x . We assume certain differentiability and other conditions on ϕ and the metric which will be stated as they are used during the calculation.

Let N be a Riemann normal neighbourhood of x with Riemann Normal Coordinates (RNC) $\{y^\mu\}$ centred on the origin x and, as before, we define spatial polar coordinates: $r = \sqrt{\sum_{i=1}^3 (y^i)^2}$, θ and φ . We also again define radial null coordinates $u = \frac{1}{\sqrt{2}}(-y^0 - r)$ and $v = \frac{1}{\sqrt{2}}(-y^0 + r)$ in N where u and v increase into the past.

We define $LC := \partial J^-(x) \cap \text{Support}(\phi)$ and assume that every point of LC lies on a unique past directed null geodesic from x . This is a strong condition – generally there will be caustics on LC – and in the final section we consider the possibility of weakening it. Each null geodesic generator of LC , $\gamma(\theta, \varphi)$, is labelled by the polar angles θ and φ and has tangent vector, $T(\theta, \varphi)$, at x with components in RNC: $T(\theta, \varphi)^\mu = \frac{1}{\sqrt{2}}(-1, \sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$.

The past pointing null tangent vectors at x come in antipodal pairs, $(T(\theta, \varphi), T(\pi - \theta, \varphi + \pi))$, such that $T(\theta, \varphi)^\mu T(\pi - \theta, \varphi + \pi)_\mu = -1$. From this we define Null Gaussian Normal Coordinates (NGNC) $\{V, U, \theta, \varphi\}$ [17] in a neighbourhood, N_{LC} , of $\partial J^-(x)$ which contains LC and is bounded away from x . The coordinates θ and ϕ are the labels of the null geodesic generators of $\partial J^-(x)$, $\gamma(\theta, \varphi)$. The coordinate V is the affine parameter along each $\gamma(\theta, \varphi)$ and is equal to v (the RNC) along the generators in the overlap of the RNC and NGNC patches. The transverse null coordinate U is the affine parameter along past pointing, ingoing null geodesics from every point on LC such that the tangent vector to the ingoing null geodesic at point p on $\gamma(\theta, \varphi)$ is the vector $T(\pi - \theta, \varphi + \pi)$ at x , parallelly transported to p along $\gamma(\theta, \varphi)$.

In calculating the integral (5), we will need to know the behaviour of the function $V(y)$, which is the volume of the causal interval between x and y .¹ For y in the near region close to x , we can use the results of Myrheim and Gibbons and Solodukhin [18, 19] to expand $V(y)$ in RNC. For the region down the light cone, we show in Appendix A that for y in N_{LC} with NGNC $\{V, U, \theta, \phi\}$ the limit of $U^{-2}V(y)$ as $U \rightarrow 0$ is finite and we denote $\lim_{U \rightarrow 0} U^{-2}V(y) = f_0(V, \theta, \varphi)$. Indeed, if the causal interval between x and y is contained in a tubular neighbourhood of null geodesic $\gamma(\theta, \varphi)$ on which there are Null Fermi Normal Coordinates (NFNC) [20] then $V(y) = U^2 f_0(V, \theta, \varphi) + U^3 G(V, U, \theta, \varphi)$, where G is a continuous function. Furthermore, f_0 is an increasing function of V and so therefore is $V(y)$, for small enough U .

Using this information we now define the regions W_i , $i = 1, 2, 3$.

Let the near region, W_1 , be a subregion of N :

$$W_1 := \{y \in N \mid 0 \leq u \leq v \leq a\} \quad (54)$$

for some $a > 0$ such that W_1 is approximately flat.

The down-the-light-cone region, W_2 , is defined by

$$W_2 := \{y \in N_{LC} \mid 0 < a'(\theta, \varphi) \leq V \leq L, 0 \leq U \leq \frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}\}, \quad (55)$$

where the cutoff L is large enough that W_2 includes the whole of LC outside W_1 . The topology of W_2 is $I \times I \times S^2$, where I is the unit interval.

¹ To avoid confusion between $V(y)$ and NGNC coordinate V we always write the volume function with its argument y .

$b > 0$ is assumed to be small enough that the entire causal interval between the origin x and any point with NGNC $(V, U, \theta, \varphi) \in W_2$ lies in a tubular neighbourhood of null geodesic, $\gamma(\theta, \varphi)$, on which Null Fermi Normal Coordinates (NFNC) exist. It is also assumed that b is small enough that the correction to $V(y)$ for $y \in W_2$ is small compared to the leading contribution so that $V(y) \approx U^{-2} f_0(V, \theta, \varphi)$. When the spacetime is flat, $u = U$ and $v = V$ on the intersection of N and N_{LC} and taking $b = a' = a$ we recover the regions defined in the previous section for Minkowski space. When there is curvature, $u \neq U$ and $v \neq V$ on the intersection of N and N_{LC} and there will be a mis-alignment between the boundaries of W_1 and W_2 for any choice of a' . However, if the normal neighbourhood N is approximately flat, then $u \approx U$ and $v \approx V$, see Figure 2. The mismatch can be made as small as we like by taking a to zero as the density ρ increases. We will keep $a' \neq a$, whilst bearing in mind that they will be almost equal.

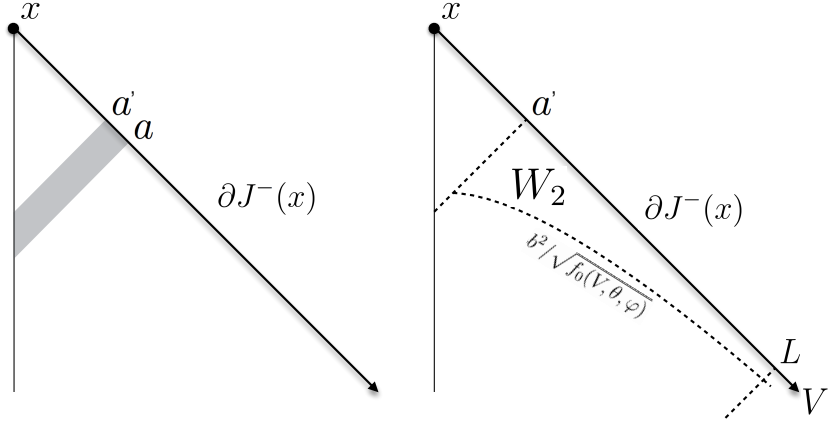


FIG. 2: The diagram on the left depicts the overlap between regions W_1 and W_2 , discussed in the text, as the shaded region. Note that here a and a' are the values of RNC and NFNC, v and V , respectively. The diagram on the right shows the integration region W_2 .

The deep chronological past, W_3 , is

$$W_3 := (\text{Supp}(\phi) \cap J^-(x)) \setminus (W_1 \cup W_2) \quad (56)$$

and is bounded away from LC .

As before we define

$$I_i = \hat{\mathcal{O}} \int_{W_i} d^4y \sqrt{-g(y)} \phi(y) e^{-\rho V(y)}, \quad (57)$$

for $i = 1, 2, 3$ so that

$$\bar{B}\phi(x) := \frac{4\sqrt{\rho}}{\sqrt{6}} \left[-\phi(x) + \rho(I_1 + I_2 + I_3) \right]. \quad (58)$$

A. Deep chronological past

Consider first

$$I_3 = \int_{W_3} d^4y \sqrt{-g} \phi(y) e^{-\rho V(y)} \times (1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3). \quad (59)$$

$V(y)$ is only zero on LC and since W_3 is bounded away from LC , $V(y)$ is bounded away from zero on W_3 : $V(y) \geq V_{\min} > 0$. So

$$\left| \int_{W_3} d^4y \sqrt{-g} \phi(y) e^{-\rho V(y)} (1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3) \right| \leq e^{-\rho V_{\min}} \int_{W_3} d^4y \sqrt{-g} \left| \phi(y) (1 - 9\rho V(y) + 8(\rho V(y))^2 - \frac{4}{3}(\rho V(y))^3) \right| \quad (60)$$

which tends to zero faster than any power of ρ^{-1} as $\rho \rightarrow \infty$.

The conditions on W_1 and W_2 given above mean that $V(y)$ attains its minimum on W_3 , on its boundary with $W_1 \cap W_2$, and its approximate value will be $V_{\min} \approx \frac{\pi}{6}a^2b^2$.

B. Down the light cone

We work in NGNC, $\{V, U, \theta, \varphi\}$, in this region.

We showed in Appendix A that in curved space the limit of $U^{-2}V(y)$ as $U \rightarrow 0$ is finite and we now assume enough differentiability of the metric so that $V(y)$ has the following expansion in W_2 :

$$V(y) = U^2 f_0(V, \theta, \varphi) + U^3 f_1(V, \theta, \varphi) + U^4 f_2(V, \theta, \varphi) + U^5 F(y). \quad (61)$$

We further assume enough differentiability of the metric and field that $\sqrt{-g(y)}$ and ϕ have the following expansions in W_2 :

$$\sqrt{-g(y)} = h_0(V, \theta, \varphi) + U h_1(V, \theta, \varphi) + U^2 h_2(V, \theta, \varphi) + U^3 H(y) \quad (62)$$

$$\phi(y) = \phi|_{U=0} + U \phi_{,U}|_{U=0} + \frac{1}{2} U^2 \phi_{,UU}|_{U=0} + U^3 \Phi(y). \quad (63)$$

The functions F , H and Φ are continuous and we have adopted a notation convention that a function denoted by a lower case letter is independent of U and a function denoted by an upper case letter may depend on U .

We will also use this expansion of the exponential factor in the integrand:

$$e^{-\rho V(y)} = e^{-\rho U^2 f_0} \left(1 - \rho U^3 (f_1 + U f_2 + U^2 F) + \frac{1}{2} \rho^2 U^6 (f_1 + U f_2 + U^2 F)^2 + \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U^{3k} (f_1 + U f_2 + U^2 F)^k \right). \quad (64)$$

We want to calculate

$$I_2 = \hat{\mathcal{O}} \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU \sqrt{-g(y)} \phi(y) e^{-\rho V(y)}. \quad (65)$$

Substituting the expansions (61) - (64) into (65) one finds three types of integrals. Integrals of the first kind, denoted by I_{21} , involve only U -independent unknown functions and do not have a factor of the infinite sum. A general such term can be written as

$$I_{21} := \hat{\mathcal{O}} \left\{ \rho^q \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU U^{n+3q} \chi(V, \theta, \varphi) e^{-\rho U^2 f_0} \right\}, \quad (66)$$

where χ is one (or a product) of the unknown functions independent of U , $q = 0, 1, 2$ and $0 \leq n \leq 4 + q$. These terms can be dealt with straightforwardly since the U -integration can be done explicitly:

$$I_{21} = \hat{\mathcal{O}} \left\{ \frac{1}{2\rho^{\frac{n+q+1}{2}}} \int d\Omega_2 \int_{a'}^L dV \frac{\chi(V, \theta, \varphi)}{f_0(V, \theta, \varphi)^{\frac{n+3q+1}{2}}} \times \left(\Gamma\left(\frac{n+3q+1}{2}\right) - \Gamma\left(\frac{n+3q+1}{2}, \rho b^4\right) \right) \right\}. \quad (67)$$

The second term is exponentially small, while first term is annihilated by $\hat{\mathcal{O}}$ for $n+q = 0, 1, 2$, and (after being multiplied by $\rho^{3/2}$) contributes a term that goes to zero in the limit for $n+q > 2$.

Integrals of the second kind involve U -dependent unknown functions (i.e. functions denoted by capital letters) and do not have a factor of the infinite sum in the integrand. These can be written as

$$I_{22} := \hat{\mathcal{O}} \left\{ \rho^q \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU U^{n+3q} \Xi(U, V, \theta, \varphi) e^{-\rho U^2 f_0} \right\}, \quad (68)$$

where $q = 0, 1, 2$ and $n = 3, \dots, 6 + 2q$. Note that for every extra factor of ρ multiplying the integral there is an extra factor of U^3 in the integrand, this ensures that I_{22} can be bounded without using properties of $\hat{\mathcal{O}}$ (c.f. term I_2 in Section II). Acting with $\hat{\mathcal{O}}$ on $\rho^q e^{-\rho U^2 f_0}$ we find

$$I_{22} = \frac{\rho^q}{3} \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU U^{n+3q} \Xi(U, V, \theta, \varphi) e^{-\rho U^2 f_0} \\ \times (3 + 11q + 12q^2 + 4q^3 - 3\rho U^2 f_0 (3 + 2q)^2 + 12\rho^2 U^4 f_0^2 (2 + q) - 4\rho^3 U^6 f_0^3), \quad (69)$$

each term of which can be bounded. We show one example here:

$$\rho^q \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU \left| U^{n+3q} \Xi(U, V, \theta, \varphi) e^{-\rho U^2 f_0} \right| \\ \leq \|\Xi\|_2 \rho^q \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU U^{n+3q} e^{-\rho U^2 f_0} \\ = \frac{\|\Xi\|_2}{2\rho^{\frac{n+3q+1}{2}}} \int d\Omega_2 \int_{a'}^L dV \frac{1}{f_0(V, \theta, \varphi)^{\frac{n+3q+1}{2}}} \\ \times \left(\Gamma\left(\frac{n+3q+1}{2}\right) - \Gamma\left(\frac{n+3q+1}{2}, \rho b^4\right) \right), \quad (70)$$

which, after being multiplied by $\rho^{3/2}$, goes to zero in the limit, since $n \geq 3$. The other terms are similar.

Finally, the remaining term in I_2 is

$$I_{23} := \hat{\mathcal{O}} \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU \sqrt{-g(y)} \phi(y) e^{-\rho U^2 f_0} \\ \times \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U^{3k} (f_1 + U f_2 + U^2 F)^k. \quad (71)$$

We will show that each term in I_{23} arising from the action of $\hat{\mathcal{O}} = (1 + 9H_1 + 8H_2 + \frac{4}{3}H_3)$ on the integrand, is $O(\rho^{-2})$, starting with the integral in I_{23} itself. First we note that

$$\left| \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U^{3k} \bar{f}^k \right| \leq \frac{\rho^3}{6} U^9 |\bar{f}|^3 e^{\rho U^3 |\bar{f}|}, \quad (72)$$

where $\bar{f} := f_1 + U f_2 + U^2 F$. Recall that in defining W_2 , b was chosen small enough that

$U^3|\bar{f}| \ll U^2 f_0$ in W_2 , so we have

$$\begin{aligned}
& \left| \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0(V, \theta, \varphi)}}} dU \sqrt{-g(y)} \phi(y) e^{-\rho U^2 f_0} \right. \\
& \quad \left. \times \sum_{k=3}^{\infty} \frac{(-\rho)^k}{k!} U^{3k} (f_1 + U f_2 + U^2 F)^k \right| \\
& \leq \frac{\rho^3}{6} \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0}}} dU \left| \sqrt{-g(y)} \phi(y) \bar{f}^3 \right| U^9 e^{-\frac{\rho}{2} U^2 f_0} \\
& \leq \frac{\|\sqrt{-g} \phi \bar{f}^3\|_2}{6} \rho^3 \int d\Omega_2 \int_{a'}^L dV \int_0^{\frac{b^2}{\sqrt{f_0}}} dU U^9 e^{-\frac{\rho}{2} U^2 f_0} \\
& = \frac{2\|\sqrt{-g} \phi \bar{f}^3\|_2}{\rho^2} \int d\Omega_2 \int_{a'}^L dV \frac{1}{f_0(V, \theta, \varphi)^5}
\end{aligned} \tag{73}$$

neglecting terms proportional to $e^{-\rho b^4/2}$. After being multiplied by $\rho^{3/2}$ this term is of order $\rho^{-1/2}$ and hence goes to zero in the limit. The terms arising from the action of H_i , $i = 1, 2, 3$ on the integrand can be bounded similarly and are also of order $\rho^{-1/2}$.

C. The near region

Now that it has been demonstrated that the contributions to the mean from the region of integration bounded away from the origin vanish in the limit, we can conclude that the result must be local since we can choose a to be arbitrarily small. The value of $\lim_{\rho \rightarrow \infty} \bar{B}\phi(x)$, if it is finite, must therefore only depend on quantities local at x . The only terms of the correct dimensions are $\square\phi(x)$ and $R\phi(x)$. This section shows that the limit is finite and that the precise linear combination is (6).

In the near region, W_1 , we work with Riemann normal coordinates $\{y^\mu\}$ centred on $x = 0$ and the usual spatial polar coordinates: $r = \sqrt{\sum_{i=1}^3 (y^i)^2}$, θ and φ , and radial null coordinates $u = \frac{1}{\sqrt{2}}(-y^0 - r)$ and $v = \frac{1}{\sqrt{2}}(-y^0 + r)$. We will show that

$$\lim_{\rho \rightarrow \infty} (\rho^{3/2} I_1 - \rho^{1/2} \phi(x)) = \frac{\sqrt{6}}{4} \left(\square\phi(x) - \frac{1}{2} R(x) \phi(x) \right), \tag{74}$$

where

$$\begin{aligned}
I_1 &= \hat{\mathcal{O}} \int_{W_1} d^4 y \sqrt{-g(y)} \phi(y) e^{-\rho V(y)} \\
&= \hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} \sqrt{-g(y)} \phi(y) e^{-\rho V(y)}.
\end{aligned} \tag{75}$$

We note that in this subsection $\sqrt{-g(y)}$ will denote the square root of (minus) the determinant of the metric *in RNC*.

In W_1 , we have expansions [19]:

$$\sqrt{-g} = 1 - \frac{1}{6}y^\mu y^\nu R_{\mu\nu}(0) + y^\mu y^\nu y^\rho T_{\mu\nu\rho}(y). \quad (76)$$

$$\begin{aligned} \phi(y) &= \phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2}y^\mu y^\nu \phi_{,\mu\nu}(0) \\ &\quad + y^\mu y^\nu y^\alpha \Psi_{\mu\nu\alpha}(y) \end{aligned} \quad (77)$$

$$\begin{aligned} V(y) &= \frac{\pi}{24}\tau^4 - \frac{\pi}{4320}\tau^6 R(0) + \frac{\pi}{720}\tau^4 y^\mu y^\nu R_{\mu\nu}(0) \\ &\quad + \tau^4 y^\mu y^\nu y^\rho S_{\mu\nu\rho}(y) = V_0(y) + \delta V(y) \end{aligned} \quad (78)$$

where $V_0(y) = \frac{\pi}{24}\tau^4 = \frac{\pi}{6}u^2v^2$ and $\delta V(y)$ is the rest, and $T_{\mu\nu\rho}(y)$, $\Psi_{\mu\nu\alpha}(y)$ and $S_{\mu\nu\rho}(y)$ are C^3 -functions.

We also use the expansion of the exponential factor,

$$e^{-\rho V(y)} = e^{-\rho V_0(y)} e^{-\rho \delta V(y)} = e^{-\rho V_0(y)} \left(1 - \rho \delta V(y) + \sum_{k=2}^{\infty} \frac{(-\rho)^k}{k!} (\delta V)^k\right). \quad (79)$$

Using (76)-(79) we expand the integrand in (75) and collect the terms in 4 groups:

$$A(y) = \phi + \frac{1}{2}y^\mu y^\nu \phi_{,\mu\nu} - \frac{1}{6}\phi y^\mu y^\nu R_{\mu\nu} + \frac{\rho\pi\tau^4}{4320}\phi(\tau^2 R - 6y^\mu y^\nu R_{\mu\nu}), \quad (80)$$

$$\begin{aligned} B(y) &= \left(1 - \frac{1}{6}y^\mu y^\nu R_{\mu\nu}\right) y^\alpha \phi_{,\alpha} - \frac{1}{12}y^\mu y^\nu y^\alpha y^\beta R_{\mu\nu} \phi_{,\alpha\beta} \\ &\quad + \rho \left(y^\alpha \phi_{,\alpha} + \frac{1}{2}y^\alpha y^\beta \phi_{,\alpha\beta} - \frac{1}{6}\phi y^\mu y^\nu R_{\mu\nu} - \frac{1}{6}y^\mu y^\nu y^\alpha R_{\mu\nu} \phi_{,\alpha} - \frac{1}{12}y^\mu y^\nu y^\alpha y^\beta R_{\mu\nu} \phi_{,\alpha\beta} \right) \\ &\quad \times \left(\frac{\pi}{4320}\tau^6 R - \frac{\pi}{720}\tau^4 y^\rho y^\sigma R_{\rho\sigma} \right) \end{aligned} \quad (81)$$

$$\begin{aligned} C(y) &= \left(1 - \frac{1}{6}y^\mu y^\nu R_{\mu\nu} + y^\mu y^\nu y^\alpha T_{\mu\nu\alpha}\right) y^\rho y^\delta y^\sigma \Psi_{\rho\delta\sigma} \\ &\quad + y^\mu y^\nu y^\alpha T_{\mu\nu\alpha} \left(\phi(0) + y^\mu \phi_{,\mu}(0) + \frac{1}{2}y^\mu y^\nu \phi_{,\mu\nu}(0) \right) \\ &\quad - \rho \left\{ \left(y^\mu y^\nu y^\alpha \Psi_{\mu\nu\alpha} - \frac{1}{6}y^\eta y^\sigma R_{\eta\sigma} y^\mu y^\nu y^\alpha \Psi_{\mu\nu\alpha} + \phi y^\mu y^\nu y^\alpha T_{\mu\nu\alpha} \right) \left(-\frac{\pi}{4320}\tau^6 R + \frac{\pi}{720}\tau^4 y^\mu y^\nu R_{\mu\nu} \right) \right. \\ &\quad + \left(1 - \frac{1}{6}y^\eta y^\sigma R_{\eta\sigma} \right) \tau^4 y^\alpha y^\beta y^\gamma S_{\alpha\beta\gamma} \left(\phi + y^\mu \phi_{,\mu}(0) + \frac{1}{2}y^\mu y^\nu \phi_{,\mu\nu}(0) + y^\alpha y^\beta y^\gamma \Psi_{\alpha\beta\gamma} \right) \\ &\quad + \phi y^\mu y^\nu y^\alpha T_{\mu\nu\alpha} \tau^4 y^\alpha y^\beta y^\gamma S_{\alpha\beta\gamma} + y^\mu y^\nu y^\alpha T_{\mu\nu\alpha} \left(y^\mu \phi_{,\mu}(0) + \frac{1}{2}y^\mu y^\nu \phi_{,\mu\nu}(0) + y^\mu y^\nu y^\alpha \Psi_{\mu\nu\alpha} \right) \\ &\quad \left. \times \left(-\frac{\pi}{4320}\tau^6 R + \frac{\pi}{720}\tau^4 y^\mu y^\nu R_{\mu\nu} + \tau^4 y^\alpha y^\beta y^\gamma S_{\alpha\beta\gamma} \right) \right\} \end{aligned} \quad (82)$$

$$D(y) = \sqrt{-g(y)} \phi(y) \sum_{k=2}^{\infty} \frac{(-\rho)^k}{k!} (\delta V)^k, \quad (83)$$

so that

$$\sqrt{-g}\phi(y)e^{-\rho V} = (A(y) + B(y) + C(y) + D(y))e^{-\rho V_0}, \quad (84)$$

and

$$I_A := \frac{1}{2}\hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 (v-u)^2 A(y) e^{-\rho V_0}, \quad (85)$$

and similarly for I_B , I_C and I_D . I_A and I_B are doable integrals involving no unknown functions. We will see that $\rho^{3/2}I_A$ gives the nonzero contributions in the limit and $\rho^{3/2}I_B$, $\rho^{3/2}I_C$ and $\rho^{3/2}I_D$ vanish in the limit.

Consider I_A . Integrating over the angular coordinates gives

$$\begin{aligned} I_A = \hat{\mathcal{O}} \int_0^a dv \int_0^v du \frac{(v-u)^2}{2} & \left(4\pi\phi + \pi(u+v)^2(\phi_{,00} - \frac{1}{3}(1 + \frac{\pi}{30}\rho u^2 v^2)\phi R_{00}) \right. \\ & \left. + \frac{\pi}{3}(v-u)^2(\phi_{,ii} - \frac{1}{3}(1 + \frac{\pi}{30}\rho u^2 v^2)\phi R_{ii}) + \frac{\pi^2}{135}\rho u^3 v^3 \phi R \right) e^{-\rho \frac{\pi}{6} u^2 v^2}. \end{aligned} \quad (86)$$

The integrals are all of the form

$$I_{m,n} := \hat{\mathcal{O}} \int_0^a dv \int_0^v du u^m v^n e^{-\frac{\pi}{6}\rho u^2 v^2}, \quad m, n \in \mathbb{N} \quad (87)$$

up to some constant factors, and can be computed explicitly. For $n \neq m$ we find

$$\begin{aligned} I_{m,n} = \frac{1}{2(m-n)}\hat{\mathcal{O}} \left[\frac{1}{\sigma^{\frac{m+n+2}{4}}} \left(\Gamma\left(\frac{m+n+2}{4}\right) - \Gamma\left(\frac{m+n+2}{4}, \sigma a^4\right) \right) \right. \\ \left. - \frac{a^{n-m}}{\sigma^{\frac{m+1}{2}}} \left(\Gamma\left(\frac{m+1}{2}\right) - \Gamma\left(\frac{m+1}{2}, \sigma a^4\right) \right) \right] \end{aligned} \quad (88)$$

where $\sigma := \rho\pi/6$, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Euler gamma function and $\Gamma(b, z) = \int_z^\infty t^{b-1} e^{-t} dt$ is the incomplete gamma function. For $n = m$,

$$I_{m,m} = \hat{\mathcal{O}} \left\{ \frac{1}{8\sigma^{\frac{m+1}{2}}} \left[\left(\ln(\sigma a^4) - \Psi\left(\frac{m+1}{2}\right) \right) \Gamma\left(\frac{m+1}{2}\right) + G_{2,3}^{3,0} \left(\sigma a^4 \left| \begin{matrix} 1, 1 \\ 0, 0, \frac{m+1}{2} \end{matrix} \right. \right) \right] \right\} \quad (89)$$

where $\psi(z) = d \ln \Gamma(z)/dz$ is the Euler ψ -function and $G_{2,3}^{3,0}$ is a G-function which decays exponentially fast in the limit $\rho \rightarrow \infty$ (see 8.36 and 9.3 in [21]).

The incomplete gamma functions are exponentially decaying in the limit, so any term in $I_{m,n}$ with $n \neq m$ and $n+m = 2, 4$ or $m = 0, 1, 2$, will be killed by the action of $\hat{\mathcal{O}}$, up to exponentially small terms, leaving the first correction to be $O(\rho^{-2})$, which does not contribute in the limit. Therefore the only terms which contribute are those proportional

to $I_{m,m}$. In I_A , the only terms proportional to $I_{m,m}$ which appear are those with $m = 1, 2$ and 4, it is then a simple exercise to show that

$$\lim_{\rho \rightarrow \infty} \frac{4}{\sqrt{6}} (-\sqrt{\rho}\phi(0) + \rho^{3/2}I_A) = (\square - \frac{1}{2}R(0))\phi(0). \quad (90)$$

Consider I_B . Again there are no unknown functions so that the integrals can be done explicitly. It is a somewhat lengthy, albeit straightforward, computation to show that the largest contribution to $\rho^{3/2}I_B$ is given by terms of order $O(\log \rho / \sqrt{\rho})$.

To help us list the leading order finite ρ corrections to the limiting value in the next section, it will be useful to note the following facts about terms in A and B . After integrating over the angular variables, all terms in A and B go like $t^i r^{2j+2}$, for $i, j \in \mathbb{N}$, where the extra power of r^2 comes from the integration measure. Now since both t and r^2 are symmetric under the exchange $u \leftrightarrow v$, when we expand $t^i r^{2j+2}$ in terms of u and v we find that for each term $u^m v^n$, $m, n \in \mathbb{N}$, in the expansion, there is a term $u^n v^m$ with the same coefficient (we deal with the degenerate case $m = n$ shortly). But, one can see from Equation (88) that the term proportional to $\sigma^{-(m+n+2)/2}$ is antisymmetric under the interchange $m \leftrightarrow n$, so that these terms always cancel. The second term in Equation (88), proportional to $a^{n-m}/\sigma^{(m+1)/2}$, is not symmetric under $m \leftrightarrow n$, and therefore contributes to the corrections with terms that are at most $O(\rho^{-1/2})$ (recall that one must multiply by an overall factor of $\rho^{3/2}$). Consider now the case where $i + 2j + 2$ is even and let $2m := i + 2j + 2$, then the expansion of $t^i r^{2j+2}$ will also contain the symmetric term $u^m v^m$. The relevant integral in this case is (89), and hence the leading order correction occurs when $m = 3, 5$, and is of order $\ln(\rho)/\sqrt{\rho}$ (note that the term with $m = 5$ is multiplied by ρ which is why it gives a correction of the same order in ρ).

Consider $C(y)$. Note first that all terms in C are of the type

$$\hat{O} \left\{ \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} y^\mu y^\nu y^\rho (y^\sigma)^m (4y^\alpha y^\beta)^n (\sigma u^2 v^2)^n \Xi_{m,n}(y) e^{-\sigma u^2 v^2} \right\}, \quad (91)$$

where m, n are non-negative integers, $\Xi_{m,n}$ are unknown functions of y labelled by m and n , and we used that $\tau^4 = 4u^2 v^2$. We can use the following identity

$$(\sigma u^2 v^2)^n e^{-\sigma u^2 v^2} = (-)^n \sigma^n \frac{\partial^n}{\partial \sigma^n} e^{-\sigma u^2 v^2} = (-)^n H_n e^{-\sigma u^2 v^2}, \quad (92)$$

together with the fact that \hat{O} and H_n commute, to rewrite these integrals as

$$(-)^n H_n \hat{O} \left\{ \int_0^a dv \int_0^v du \int d\Omega_2 \frac{(v-u)^2}{2} y^\mu y^\nu y^\rho (y^\sigma)^m (4y^\alpha y^\beta)^n \Xi_{m,n}(y) e^{-\sigma u^2 v^2} \right\}. \quad (93)$$

Ignoring the operator $(-)^n H_n$, which does not change the overall power of σ coming from integration, we see that the remaining integral is the same as equation (37) appearing in Section II (except that here $m+n \geq 3$ while in the flat section we had $m+n = 3$). We can therefore apply the analysis used in Section II to deal with these integrals. Doing this we find that the leading corrections are of order $\rho^{-1/4}$.

Again, it is useful for the next section to note that the leading order corrections coming from terms in C come from equations (48), and (50) with $i+j = 2$. Since these terms do not involve any powers of a we can determine which terms in C give these contributions on dimensional grounds. We find that the leading order contributions come from $\|\Psi_{\mu\nu\alpha}\|_1$ and $\|T_{\mu\nu\alpha}\|_1\phi$ and are of order $\rho^{-1/4}$.

Finally we deal with I_D ,

$$I_D = \hat{\mathcal{O}} \int_0^a dv \int_0^v du \int d\Omega_2 f(y) \sum_{k=2}^{\infty} \frac{(-\rho)^k}{k!} (\delta V)^k e^{-\rho V_0} \quad (94)$$

where $f(y) := (v-u)^2 \sqrt{-g(y)} \phi(y)/2$. Let us write $\delta V = \tau^4 \xi(y)$, $\xi(y) = y^\mu y^\nu \chi_{\mu\nu}(y)$, where by assumption $\|\xi\|_1 \ll \frac{\pi}{24} < 1$ in W_1 , and leave out the angular integration to be done at the end, then

$$\begin{aligned} & \hat{\mathcal{O}} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} \frac{1}{k!} f(y) \xi^k (-\rho)^k \tau^{4k} e^{-\rho \frac{\pi}{24} u^2 v^2} \\ &= \hat{\mathcal{O}} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} \frac{1}{k!} f(y) \xi^k H_k e^{-\rho \frac{\pi}{24} u^2 v^2} \\ &= \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} H_k \left\{ \frac{1}{k!} f(y) \xi^k \hat{\mathcal{O}} e^{-\rho \frac{\pi}{24} u^2 v^2} \right\} \\ &= -\frac{1}{6} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} H_k \left\{ \frac{1}{k!} u^3 \frac{\partial^3}{\partial u^3} (f(y) \xi^k) e^{-\rho \frac{\pi}{24} u^2 v^2} \right\} \\ &\quad + \frac{1}{3} \int_0^a dv \sum_{k=2}^{\infty} H_k \left\{ 2^k v^{6+2k} \tilde{h} \tilde{\chi}_{00}^k e^{-\rho v^4} \right\}, \end{aligned} \quad (95)$$

where $h = f/(v-u)^2$, a tilde means set $u = v$, $2v^2 \tilde{\chi}_{00} = \tilde{\xi}$, $H_k = \rho^k \partial^k / \partial \rho^k$, the third equality used $[\hat{\mathcal{O}}, H_k] = 0$, the final equality was obtained by integrating by parts three times and using the properties of $\hat{\mathcal{O}}$ (c.f. equations (39)-(47)), and we could exchange the order of summation and integration because the partial sums are uniformly integrable.

Let us now consider the first term in the above expression. Acting with H_k we find

$$\begin{aligned} & -\frac{1}{6} \int_0^a dv \int_0^v du \sum_{k=2}^{\infty} \frac{(-\rho)^k}{k!} \tau^{4k} u^3 \frac{\partial^3}{\partial u^3} (f(y) \xi^k) e^{-\rho \frac{\pi}{6} u^2 v^2} \\ & = \frac{\rho^2}{6} \int_0^a dv \int_0^v du \sum_{k=0}^{\infty} \frac{(-\rho)^k}{(k+2)!} \tau^{4k} \tau^8 u^3 \frac{\partial^3}{\partial u^3} [f(y) \xi^{k+2}] e^{-\rho \frac{\pi}{6} u^2 v^2} \end{aligned} \quad (96)$$

Now

$$\begin{aligned} \frac{\partial^3}{\partial u^3} (f \xi^{k+2}) &= [f''' \xi^2 + 3(k+2) f'' \xi \xi' + 3(k+2)(k+1) f' (\xi')^2 + 3(k+2) f' \xi \xi'' \\ &+ 3(k+2)(k+1) f \xi' \xi'' + (k+2) f \xi \xi'''] \xi^k + k(k+2)(k+1) f (\xi')^3 \xi^{k-1}, \end{aligned} \quad (97)$$

where $'$ denotes differentiation with respect to u . Note that since $f = (v-u)^2 \sqrt{-g(y)} \phi(y)/2$ and $\xi = y^\mu y^\nu \chi_{\mu\nu}(y)$, each of the terms multiplying ξ^k is given by $u^i v^j$ with $3 \leq i+j \leq 6$, up to an unknown function; while the term multiplying ξ^{k-1} is only nonzero for $k \geq 1$ and is given by $u^i v^j$, $5 \leq i+j \leq 8$, up to an unknown function. In light of this we rewrite (97) as

$$\sum_{3 \leq i+j \leq 6} u^i v^j \Xi_{i,j}(y) \xi(y)^k + \sum_{5 \leq i+j \leq 8} u^i v^j \Phi_{i,j}(y) \xi(y)^{k-1} \quad (98)$$

where $\Xi_{i,j}$ and $\Phi_{i,j}$ are unknown functions labelled by i, j , and $\Phi_{i,j} = 0$ when $k = 0$.

Integral (96) can therefore be bounded by a sum of two types of terms (ignoring overall constant factors):

$$\left| \rho^2 \int_0^a dv \int_0^v du \sum_{k=0}^{\infty} \frac{(-\rho)^k}{(k+2)!} \tau^{4k} \tau^8 u^3 u^i v^j \Xi_{i,j} \xi(y)^k e^{-\rho \frac{\pi}{6} u^2 v^2} \right|, \quad (99)$$

where $3 \leq i+j \leq 6$, and

$$\left| \rho^3 \int_0^a dv \int_0^v du \sum_{k=0}^{\infty} \frac{(-\rho)^k}{(k+3)!} \tau^{4k} \tau^{12} u^3 u^i v^j \Phi_{i,j} \xi(y)^k e^{-\rho \frac{\pi}{6} u^2 v^2} \right| \quad (100)$$

where $5 \leq i+j \leq 8$, corresponding to the first and second terms in (98) respectively.

A general term in (99) is bounded by

$$I_{i,j} := \|\Xi_{i,j}\|_1 \rho^2 \int_0^a dv \int_0^v du 16 u^3 u^4 v^4 u^i v^j \left| \sum_{k=0}^{\infty} \frac{\xi^k}{(k+2)!} (-\rho)^k \tau^{4k} \right| e^{-\rho \frac{\pi}{6} u^2 v^2}, \quad (101)$$

where $3 \leq i+j \leq 6$. We can bound the infinite sum

$$\left| \sum_{k=0}^{\infty} \frac{\xi^k}{(k+2)!} (-\rho)^k \tau^{4k} \right| \leq \frac{1}{2} e^{\rho \frac{\pi}{48} \tau^4} \quad (102)$$

$\forall y \in W_1$, and we have chosen $\|\xi\|_1 \leq \pi/48$. Substituting this back into (101) gives

$$I_{i,j} \leq \|\Xi_{i,j}\|_1 8\rho^2 \int_0^a dv \int_0^v du u^{7+i} v^{4+j} e^{-\rho \frac{\pi}{12} u^2 v^2} \\ = \left\{ \begin{array}{ll} \frac{4\|\Xi_{i,j}\|_1}{(i-j+3)} \left(\frac{\Gamma((13+i+j)/4)}{\sigma^{(5+i+j)/4}} - \frac{a^{j-i-3}\Gamma(4+i/2)}{\sigma^{2+i/2}} \right), & i-j+3 \neq 0 \\ \frac{\|\Xi_{i,j}\|_1}{\sigma^{2+i/2}} (\ln(\sigma a^4) - \Psi(4+i/2)) \Gamma(4+i/2), & i-j+3 = 0 \end{array} \right\}, \quad (103)$$

up to exponentially small terms, where $\sigma = \pi\rho/12$. Corrections, after multiplication by $\rho^{3/2}$, therefore go to zero at least as fast as $\ln(\rho)/\sqrt{\rho}$. Applying the same analysis to (100) again gives corrections of order $\ln(\rho)/\sqrt{\rho}$ (the extra power of ρ is compensated by an extra factor of τ^4 in the integrand).

Let us now turn to the second term in (95) given by

$$\frac{1}{3} \int_0^a dv \sum_{k=2}^{\infty} H_k 2^k v^{6+2k} \tilde{h} \tilde{\chi}_{00}^k e^{-\frac{\pi}{24} \rho v^4}. \quad (104)$$

We will bound this term as we did the previous one:

$$\left| \sum_{k=2}^{\infty} \int_0^a dv 2^k v^{6+2k} \tilde{h} \tilde{\chi}_{00}^k e^{-\frac{\pi}{24} \rho v^4} (-\rho)^k v^{4k} \right| \\ \leq 4\rho^2 \|\tilde{h}\|_1 \|\tilde{\chi}_{00}^2\|_1 \int_0^a dv v^6 v^{12} e^{-\frac{\pi}{6} \rho v^4} \left| \sum_{k=0}^{\infty} \frac{(2v^2 \tilde{\chi}_{00})^k}{(k+2)!} (-\rho)^k v^{4k} \right| \\ \leq 2\rho^2 \|\tilde{h}\|_1 \|\tilde{\chi}_{00}^2\|_1 \int_0^a dv v^{18} e^{-\rho \frac{\pi}{12} v^4}, \quad (105)$$

Again this last integral gives a contribution of order $\rho^{-5/4}$, after multiplication by $\rho^{3/2}$.

D. Finite ρ corrections

Similarly to what was done in flat spacetime, we ask how $\bar{B}\phi(x)$ behaves when $\rho = l^{-4}$ is large but finite. Unlike in flat spacetime however, we will only estimate corrections in the “near region”, W_1 , since we lack an explicit expression for the expansion of the volume of long skinny intervals “down the light cone”, W_2 , and therefore any correction in W_2 can only be given in terms of integrals of unknown functions, and thus are not very enlightening.

As was noted in the previous section, terms in A and B give contributions of order $\ln(\rho)/\sqrt{\rho}$, and terms in C and D give leading contributions of order $\rho^{-1/4}$ and $\ln(\rho)/\sqrt{\rho}$ respectively. In particular we find that $\bar{B}\phi$ is a good approximation to $(\square - R/2)\phi$ at finite

ρ whenever there exists a coordinate frame and a length scale a such that $\rho a^4 \gg 1$, and such that the following conditions in that frame hold:

$$l^2 a^{n-3} X_n(0), l^2 \ln(a/l) X_3(0) \ll (\square - \frac{1}{2}R)\phi(0), \quad (106)$$

$n = 0, \dots, 5$, and $X_n(0)$ is a product of terms evaluated at the origin of dimension $[X] = 1/\text{Length}^{n+2}$. E.g. for $n = 2$, $X_2 \in \{\phi_{,0}R_{00}, \phi_{,i}R_{0i}, \phi_{,0}R\}$.

Corrections coming from terms C and D can only be bounded, because of the presence of unknown functions. We find for C -terms

$$l \|\Psi_{\mu\nu\alpha}\|_1, l \|T_{\mu\nu\alpha}\|_1 \phi, l^2 a^{n-1} Y_n \ll (\square - \frac{1}{2}R)\phi(0), \quad (107)$$

$n = 0, \dots, 9$ and Y_n is the uniform norm over W_1 of products, and u -derivatives of products, of unknown functions, times quantities evaluated at the origin, and is of dimension $[Y_n] = 1/\text{Length}^{n+4}$. E.g. if we let $\|\Psi\|_1 := \|\Psi_{\mu\nu\alpha}\|_1$, $\|T\|_1 := \|T_{\mu\nu\alpha}\|_1$ and $\|S\|_1 := \|S_{\mu\nu\alpha}\|_1$ for any μ, ν, α , then

$$Y_2 \in \{\|\Psi''\|_1, \phi\|T''\|_1, \phi\|S''\|_1, \phi_{,\mu}\|T'\|_1, \phi_{,\mu}\|S'\|_1, \phi_{,\mu\nu}\|T\|_1, \phi_{,\mu\nu}\|S\|_1, \\ R_{\mu\nu}\|\Psi\|_1, \phi R_{\mu\nu}\|T\|_1, \phi R_{\mu\nu}\|S\|_1, \phi R\|T\|_1, R\|\Psi\|_1\}.$$

While D -terms contribute

$$l^2 \ln(a/l) \|\Xi_{0,3}\|_1, l^2 \|\Xi_{1,2}\|_1, l^2 \|\Xi_{2,1}\|_1, l^2 \|\Xi_{3,0}\|_1, l^2 a^n \|\Xi_{0,3+n}\|_1 \ll (\square - \frac{1}{2}R)\phi(0), \quad (108)$$

$n = 0, 1, 2, 3$, and

$$l^2 a^m \|\Phi_{0,3+m}\|_1 \ll (\square - \frac{1}{2}R)\phi(0), \quad (109)$$

$m = 2, \dots, 6$.

IV. DISCUSSION

In [11, 12], causal set d'Alembertians were defined for dimensions $d = 3$ and $d > 4$, and it was shown that if the mean of these d'Alembertians has a local limit as $\rho \rightarrow \infty$ then that limit will be $\square - R/2$ in all dimensions. We expect the argument for the existence of the local limit under certain conditions given above for $d = 4$, to be able to be generalised for $d = 3$ and $d > 4$. In two dimensions, the original case looked at by Sorkin, the conformal flatness of spacetime should make the proof substantially easier.

We have been ignoring the important question of the fluctuations about the mean. These fluctuations are large and grow with the sprinkling density. In order to tame these fluctuations, Sorkin introduced, in two dimensions, for each fixed physical non-locality length scale, $l_k \geq l$, a causal set operator, $B_k^{(2)}$ [7], whose mean over sprinklings at density ρ does not depend on ρ but is equal to the mean of $B^{(2)}$ with ρ replaced by l_k^{-2} . This was extended to four dimensions [10]:

$$B_k^{(4)}\phi(x) = \frac{4}{\sqrt{6}l_k^2} \left[-\phi(x) + \epsilon \sum_{y \prec x} f(n(x, y), \epsilon) \phi(y) \right], \quad (110)$$

where $\epsilon = (l/l_k)^4$ and

$$f(n, \epsilon) = (1 - \epsilon)^n \left[1 - \frac{9\epsilon n}{1 - \epsilon} + \frac{8\epsilon^2 n!}{(n - 2)!(1 - \epsilon)^2} - \frac{4\epsilon^3 n!}{3(n - 3)!(1 - \epsilon)^3} \right], \quad (111)$$

and also to all other dimensions [11, 12]. In each dimension, d , the mean of $B_k^{(d)}\phi$ over sprinkling at density $\rho = l^{-d}$ takes the same form as for the original d'Alembertian but with the discreteness scale ρ replaced by $\rho_k = l_k^{-d}$. So, the mean, $\bar{B}_k^{(4)}\phi$, of $B_k^{(4)}\phi$ is given by (5) with ρ replaced by l_k^{-4} . Thus, results about \bar{B} at finite ρ can be applied to \bar{B}_k . Simulations of $B_k^{(d)}$ for a selection of test functions in 2, 3 and 4 dimensional flat space indicate that its fluctuations do die away as $l \rightarrow 0$ [7, 10, 11] but this remains to be more thoroughly tested.

In calculating the limit of the mean of the causal set d'Alembertian in curved space, we made the assumption that between x and every point of $\partial J^-(x)$ there is a unique null geodesic. This is a strong assumption and it is possible that it can be weakened. The assumption is made so that $\partial J^-(x)$ can be treated as a null geodesic congruence, guaranteeing the existence of Null Gaussian Normal Coordinates in a neighbourhood of $\partial J^-(x)$ which allows the integral to be performed. When the assumption fails and there are caustics on $\partial J^-(x)$ it is nevertheless the case, in a globally hyperbolic spacetime, that every point on $\partial J^-(x)$ lies on at least one null geodesic from x . Moreover, the set of points on $\partial J^-(x)$ which are not connected to x by a single null geodesic consists of those points that lie on caustics and is a set of measure zero in $\partial J^-(x)$. It might be possible to construct a proof in the general case by covering the region of integration down the light cone with appropriate finite collections of subregions in which Null Gaussian Normal Coordinates can be constructed. If the integral can be performed in each subregion and shown to be equal to zero in the limit, one might be able to argue that the whole integral also tends to zero.

The importance of such a result is clouded, however, by the fact that we have not been able to interpret physically the conditions under which the finite ρ corrections to the mean from the down-the-light-cone region are small. This is because in order to estimate the corrections one needs an explicit form for the volume of causal intervals which hug the past light cone, the so-called *long skinny intervals*. These intervals have small volume but are not necessarily approximately flat. For an interval to be approximately flat, there must exist a Riemann normal neighbourhood with RNC $\{x^\mu\}$ in which the metric in the whole interval is approximately the Minkowski metric and the interval is upright *i.e.* the timelike geodesic between the two endpoints has tangent vector $\frac{\partial}{\partial x^0}$. We lack an explicit expansion of $V(y)$ for a long skinny interval in the NFNC U : for example, we do not know whether $f_0(V, \theta, \varphi)$ can be expressed simply in terms of the curvature components along the light cone.

We did show that if the support of ϕ is contained in an approximately flat Riemann normal neighbourhood (what we have called the near region in our calculation) then $\bar{B}\phi$ is a good approximation to $(\square - R/2)\phi$ at finite density ρ if conditions in Section III D hold. In fact, not all of these conditions are independent of each other, and most can actually be satisfied if the following physical conditions hold: $a \gg l$, $\lambda \gg l$, $r \gg a$. Thus $\bar{B}\phi(x)$ is well approximated by $(\square - R/2)\phi(x)$ when there exists a frame in which the characteristic length scales on which ϕ and the metric vary are both large compared to l .

The value of the continuum limit of the mean of the causal set d'Alembertian was used to propose a causal set action in $d = 2, 4$ [10] and in other dimensions [11]. Actions derived from B_k have also been defined.

Our results suggest that the action in $d = 4$ evaluated on a causal set that is a sprinkling of an approximately flat region of a four dimensional spacetime will be approximately equal to the Einstein-Hilbert (EH) action of the spacetime – if the fluctuations around the mean are small. However, it is possible that – due to the long skinny neighbourhood issue – the action evaluated on a causal set that is a sprinkling into a spacetime which is large compared to the scale set by the curvature is *not* close to the Einstein-Hilbert action. If this turns out to be the case, we would have a discrete action that is close to the EH action for regions small compared to any radius of curvature, but starts to deviate from it as the size of the region L approaches the curvature length scale set by R , which here stands for any curvature component. If the proposed causal set action is relevant in a continuum regime of full quantum gravity, this might indicate that the continuum approximation to quantum

gravity is General Relativity when $RL^2 \ll 1$ but starts to differ from General Relativity when $RL^2 \sim 1$.

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Appendix A

Consider a point y in N_{LC} with Null Gaussian Normal Coordinates (U, V, θ, ϕ) and the volume, $V(y)$ of the causal interval, $\text{Int}(x, y)$, between x and y .

We use the null geodesic $\gamma(\theta, \varphi)$ defined in the text to define Null Fermi Normal Coordinates (NFNC) (x^+, x^-, x^1, x^2) associated to $\gamma(\theta, \varphi)$ [20]. They are defined using a pseudo-orthonormal tetrad at x , $\{E_+, E_-, E_1, E_2\}$ where $E_+ = T(\theta, \varphi)$, $E_- = T(\pi - \theta, \varphi + \pi)$ and E_1 and E_2 are spacelike unit vectors, orthogonal to each other and to both $T(\theta, \varphi)$ and $T(\pi - \theta, \varphi + \pi)$. The affine parameter along $\gamma(\theta, \varphi)$ is x^+ . This tetrad is parallel transported along $\gamma(\theta, \varphi)$: $\{E_+(x^+), E_-(x^+), E_1(x^+), E_2(x^+)\}$. The NFNC are normal coordinates defined by the spray of geodesics emanating from each point along $\gamma(\theta, \varphi)$ with tangent vectors lying in the subspace of the tangent space spanned by $\{E_-(x^+), E_1(x^+), E_2(x^+)\}$.

In these NFNC the point y has coordinates $(x^+ = V, x^- = U, x^1 = 0, x^2 = 0)$. Indeed, outside any neighbourhood of x , the 2-dimensional surface defined in NGNC by θ constant and ϕ constant is the same surface as that defined in the NFNC (associated with $\gamma(\theta, \varphi)$) by $x^1 = x^2 = 0$.

In NFNC the metric to quadratic order is [20]

$$\begin{aligned}
ds^2 = & -2dx^+dx^- + \delta_{ab}dx^adx^b \\
& - \left[R_{+\bar{a}+\bar{b}}(x^+) x^{\bar{a}}x^{\bar{b}}(dx^+)^2 + \frac{4}{3}R_{+\bar{b}\bar{a}\bar{c}}(x^+)x^{\bar{b}}x^{\bar{c}}(dx^+dx^{\bar{a}}) + \frac{1}{3}R_{\bar{a}\bar{c}\bar{b}\bar{d}}(x^+)x^{\bar{c}}x^{\bar{d}}(dx^{\bar{a}}dx^{\bar{b}}) \right] \\
& + \mathcal{O}(x^{\bar{a}}x^{\bar{b}}x^{\bar{c}})
\end{aligned} \tag{A1}$$

where all the curvature components are evaluated on the null geodesic, the barred indices \bar{a} *etc.* run over the three transverse directions $- , 1, 2$ and unbarred over the spatial transverse directions 1 and 2 only.

Fixing y with its NGNC $\{U, V, \theta, \phi\}$, we assume that the causal interval $\text{Int}(x, y)$ between x and y lies in the tubular neighbourhood of $\gamma(\theta, \varphi)$ on which the NFNC are defined. We want to evaluate

$$V(y) = \int_{\text{Int}(x, y)} dx^+dx^-dx^1dx^2\sqrt{-g(x^+, x^-, x^1, x^2)}, \tag{A2}$$

the volume of $\text{Int}(x, y)$ as an expansion in U as $U \rightarrow 0$. In other words we want to consider the limit as the interval tends to the segment of the null geodesic $\gamma(\theta, \varphi)$ between x and the point with NGNC $\{0, V, \theta, \phi\}$. This is related to the Penrose limit.

Rescaling the coordinates $z^- = x^-/U$, $z^+ = x^+$ and $z^a = x^a/\sqrt{U}$, in the limit $U \rightarrow 0$ the metric becomes

$$ds^2 = U \left(2dz^+dz^- + \delta_{ab}dz^adz^b + R_{+a+b}(z^+)z^az^b(dz^+)^2 \right) + O(U^{\frac{3}{2}}) \tag{A3}$$

and the next terms proportional to $U^{\frac{3}{2}}$ and U^2 in the expansion can be found in Appendix A of reference [20]. Using this one can show that $\sqrt{-g(z^+, z^-, z^1, z^2)} = U^2[1 + Uf(U, z^+, z^-, z^1, z^2)]$, where f is a differentiable function of U .

$$V(y) = \int_{\text{Int}(x, y)} dz^+dz^-dz^1dz^2\sqrt{-g(z^+, z^-, z^1, z^2)} \tag{A4}$$

$$= U^2 \int_{\text{Int}(x, y)} dz^+dz^-dz^1dz^2[1 + Uf(U, z^+, z^-, z^1, z^2)]. \tag{A5}$$

The interval $\text{Int}(x, y)$ is defined by the causal structure of the metric and is the same for ds^2 as for the conformally rescaled metric

$$\tilde{ds}^2 = U^{-1}ds^2 = 2dz^+dz^- + \delta_{ab}dz^adz^b + R_{+a+b}(z^+)z^az^b(dz^+)^2 + O(U^{\frac{1}{2}}). \tag{A6}$$

In the $U \rightarrow 0$ limit, therefore, the integral

$$\int_{\text{Int}(x,y)} dz^+ dz^- dz^1 dz^2 \quad (\text{A7})$$

tends to the volume of the causal interval between the origin and the point with coordinates $(z^+ = V, z^- = 1, z^1 = 0, z^2 = 0)$ in the Penrose limit metric

$$ds^2 = 2dz^+ dz^- + \delta_{ab} dz^a dz^b + R_{+a+b}(z^+) z^a z^b (dz^+)^2. \quad (\text{A8})$$

Note that R_{+a+b} are the curvature components of the original metric along $\gamma(\theta, \phi)$ in the original, unscaled NFNC. We denote this limit volume by $f_0(V, \theta, \phi)$. It is an open question whether f_0 can be expressed more concretely in terms of (integrals of?) the curvature components along $\gamma(\theta, \phi)$.

We conclude that $U^{-2}V(y) \rightarrow f_0(V, \theta, \phi)$ as $U \rightarrow 0$ and so $V(y) = U^2 f_0(V, \theta, \phi) + U^3 G(V, U, \theta, \phi)$ with G a continuous function of U . In fact, we need to assume more differentiability than this for $V(y)$ for the proof but the crucial fact established here is the U^2 behaviour of the leading term in the expansion.

We can also show that f_0 is a monotonic increasing function of V . Consider two points p and p' in the Penrose limiting geometry (A8) with coordinates $(z^+ = V, z^- = 1, z^1 = 0, z^2 = 0)$ and $(z^+ = V', z^- = 1, z^1 = 0, z^2 = 0)$ respectively, where $V < V'$. There is a future pointing null geodesic, along which $z^- = 1, z^1 = 0, z^2 = 0$, from p' to p and so the causal interval from p' to x contains the causal interval from p to x . Then, if U is small enough, it follows that $V(y)$ is monotonic increasing in V .